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THE DYNAMICS OF A LAGRANGE TOP WITH A VIBRATING SUSPENSION POINT†

O. V. KHOLOSTOVA

Moscow

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The motion of a Lagrange top, whose suspension point performs high-frequency vertical harmonic oscillations of small amplitude, is considered. The angular velocities of the natural rotation of the top and of the rotation of its axis of symmetry around the vertical are assumed to be small. It is well known that, in the case of a classical Lagrange top with a fixed suspension point, for any values of the parameters of the problem (the values of the constants of cyclical integrals) there is a unique regular precession of the top. When the suspension point vibrates the following result is established, which has no analogues in the classical problem: regions are distinguished in the plane of those parameters in which, for any position of the centre of gravity of the top on the axis of symmetry, there is a unique periodic motion of the top (with a period equal to the period of oscillations of the suspension point), close to regular precession, and also regions in which, depending on the position of the centre of gravity, there can be one or three such motions. A rigorous solution of the problem of the stability of these motions of the top is given using the methods of the KAM theory. © 2000 Elsevier Science Ltd. All rights reserved.

This paper is a development of the results obtained in [1], where the problem of the periodic motions of a spherical pendulum with a vibrating suspension point is solved with assumptions similar to those used here.

A number of investigations have been devoted to different aspects of the problem of the dynamics of a rigid symmetrical body with a vibrating suspension point: the motion of a rapidly rotating symmetrical and close to symmetrical gyroscope when there are vertical vibrations of the suspension point has been investigated in [2, 3], the behaviour of a Lagrange top when the suspension point performs harmonic oscillations in a horizontal plane was considered in [4], the motion of a viscoelastic rigid body with a moving base was investigated in [5], and the rotation of a Lagrange top when there are random oscillations of the point of support was considered in [6].

1. FORMULATION OF THE PROBLEM. CONVERSION OF THE HAMILTON FUNCTION

Consider a dynamic symmetrical rigid body moving in a uniform gravity field around a fixed point O. Suppose the centre of mass of the body lies on its dynamic-symmetry axis. This rigid body is called a Lagrange top; its motion was investigated in detail in [7–9].

We will assume that the point O executes vertical motion in accordance with the law $O_*O = \xi(T)$ about a certain fixed point O. Suppose $OXYZ$ is a system of coordinates moving translationally in absolute space (the OZ axis is directed vertically upwards) and $Oxyz$ is a system of coordinates, rigidly attached to the body, whose axes coincide with the principal axes of inertia of the body for the point O, where the Oz axis is directed along the dynamic-symmetry axis, and the centre of mass G of the body lies on the positive semiaxis $Oz (OG = z_G, z_G > 0)$. We will specify the orientation of the system of coordinates $Oxyz$ with respect to $OXYZ$ using the Euler angles.

The kinetic energy of the body is given by the expression

$$T = \frac{1}{2} m\mathbf{v}_O^2 + m\mathbf{v}_O \cdot \mathbf{v}_{G_{\text{rel}}} + \frac{1}{2}[A(p^2+q^2)+Cr^2] \tag{1.1}$$

where m is the mass of the body, A and C are the equatorial and axial moments of inertia respectively, $\mathbf{v}_o$ is the velocity of the point O, $\mathbf{v}_{G\text{rel}} = \boldsymbol{\omega} \times \overrightarrow{OG}$ is the velocity of the point G in the system of coordinates $OXYZ$, and ω is the vector of the absolute angular velocity of rotation of the body, having projections p, q and r in the attached system of coordinates.

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In projections onto the $Oxyz$ axes we have $\vec{OG} = (0, 0, z_G)^T$, $\mathbf{v}_{G\mathrm{rel}} = (qz_G, -pz_G, 0)^T$, $\mathbf{v}_O = \dot{\xi}\mathbf{n}$, where $\mathbf{n} = (\sin\theta\sin\varphi, \sin\theta, \cos\varphi, \cos\theta)^T$ is the unit vector of the vertical axis OZ.

From (1.1) and Euler's kinematic equations we have the following expression for the kinetic energy of the body

$$T = \frac{1}{2}m\dot{\xi}^2 - mz_G\dot{\xi}\dot{\theta}\sin\theta + \frac{1}{2}A(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}C(\dot{\psi}\cos\theta + \dot{\varphi})^2 \tag{1.2}$$

The potential energy of the body can be calculated from the formula

$$\Pi = mgz_G\cos\theta + mg\xi(t) \tag{1.3}$$

It follows from (1.2) and (1.3) that the coordinates ψ and φ are cyclical, and the momenta corresponding to them are the same as in the case of the motion of a Lagrange top with a fixed point O

$$p_\psi = A\dot{\psi}\sin^2\theta + C(\dot{\psi}\cos\theta + \dot{\varphi})\cos\theta \tag{1.4}$$

$$p_\varphi = C(\dot{\psi}\cos\theta + \dot{\varphi})$$

We will introduce the notation $p_\psi = Aa, p_\varphi = Ab$ for the constant quantities p_ψ and p_φ (where a and b are constants). We then have from (1.4)

$$\dot{\psi} = \frac{a - b\cos\theta}{\sin^2\theta}, \quad \dot{\varphi} = \frac{A}{C}b - \frac{(a - b\cos\theta)\cos\theta}{\sin^2\theta} \tag{1.5}$$

The momentum p_θ, corresponding to the positional coordinate θ, depends on the motion of the point O and given by the equation

$$p_\theta = A\dot{\theta} - mz_G\dot{\xi}\sin\theta \tag{1.6}$$

From (1.2), (1.3), (1.5) and (1.6) we obtain the following expression for the Hamilton function (unimportant terms which are functions of time or are constant are omitted)

$$H = \frac{A(a - b\cos\theta)^2}{2\sin^2\theta} + \frac{(p_\theta + mz_G\dot{\xi}\sin\theta)^2}{2A} + mgz_G\cos\theta \tag{1.7}$$

The Hamiltonian (1.7) corresponds to a system with one degree of freedom with generalized coordinate θ.

We will further assume that $\xi(t) = a_*\cos\Omega t$. We will introduce the dimensionless time $\tau = \Omega t$ and dimensionless parameters of the problem and the momentum p_θ by the formulae $a = \Omega a'$, $b = \Omega b'$, $p_\theta = A\Omega p'_\theta$. Hamiltonian (1.7) can then be rewritten in the form

$$H' = \frac{(a' - b'\cos\theta)^2}{2\sin^2\theta} + \frac{1}{2}(p'_\theta - c\sin\tau\sin\theta)^2 + d\cos\theta, \tag{1.8}$$

where

$$c = \frac{mz_Ga_*}{A}, \quad d = \frac{mgz_G}{A\Omega^2} \quad (c > 0,\ d > 0)$$

We will further assume that: (1) the amplitude a_* of the vibrations of the point O is small compared with the characteristic dimension of the body, (2) the natural frequency $\sqrt{(g/l)}$ ($l = A/(mz_G)$ is the reduced length) of small oscillations of the body as a physical pendulum (when $a' = b' = 0$) in the neighbourhood of stable equilibrium $\theta = \pi$ is much less than the frequency Ω of the vibrations of the point O, and (3) the quantities a' and b', representing the angular velocities of natural rotation of the body $\dot{\varphi}$ and the rotation of its axis of symmetry around the vertical $\dot{\psi}$, are small. Taking these assumptions into account, we have that

$$c = a_*/l = \varepsilon^2 \ (0 < \varepsilon \ll 1), \quad d = g/(\Omega^2 l) = \varepsilon^4\gamma \ (\gamma > 0), \quad a' = \varepsilon^2\alpha, \quad b' = \varepsilon^2\beta$$

The parameters α and β can be taken to be arbitrary quantities. We will further assume that

$\alpha^2 \neq \beta^2$. The case $\alpha^2 = \beta^2$, when the axis of symmetry of the top can occupy the vertical position ($\theta = 0$ or $\theta = \pi$) requires a special consideration.

Making the change of variables $\theta, p'_\theta \to x, X$ in the Hamiltonian (1.8) using the formulae $\theta = x$, $p'_\theta \to \varepsilon X$, we can rewrite it, taking the notation employed into account, in the form

$$H' = H_0 + \varepsilon H_1 + \frac{1}{2!}\varepsilon^2 H_2 + \frac{1}{3!}\varepsilon^3 H_3 \tag{1.9}$$

$$H_0 = 0, \quad H_1 = \frac{1}{2}X^2, \quad H_2 = -2X\sin\tau\sin x$$

$$H_3 = 3\left[\sin^2\tau\sin^2 x + 2\gamma\cos x + \frac{(\alpha - \beta\cos x)^2}{\sin^2 x}\right]$$

We will further carry out the canonical transformation $x, X \to q, p$, 2π-periodic with respect to τ, such that the new Hamilton function does not contain the time τ in terms up to the third order inclusive in ε. We obtain its transformation using the Depry–Hori method [10].

The new Hamiltonian $K(q, p, \tau)$ must have the following structure

$$K = K_0 + \varepsilon K_1 + \frac{1}{2!}\varepsilon^2 K_2 + \frac{1}{3!}\varepsilon^3 K_3 + O(\varepsilon^4) \tag{1.10}$$

where $K_0 = 0$ and the functions K_1, K_2 and K_3 are found from the relation [10]

$$K_1 = H_1 - \partial W_1 / \partial t, \quad K_2 = H_2 + L_1 H_1 + K_{1,1} - \partial W_2 / \partial t$$

$$K_3 = H_3 + L_1 H_2 + 2L_2 H_1 + 2K_{1,2} + K_{2,1} - \partial W_3 / \partial t$$

Here $L_j = (f, W_j)$ is the Poisson bracket of the functions f and W_j, $K_{1,1} = L_1 K_1$, $K_{1,2} = L_1 K_2$, $K_2, 1 = L_2 K_1 - L_1 K_{1,1}$, while the functions $W_i(q, p, \tau)$ ($i = 1, 2, 3$) are chosen so that the quantities K_i ($i = 1, 2, 3$) do not contain τ. Calculations show that

$$W_1 = 0, \quad W_2 = 2p\cos\tau\sin q, \quad W_3 = -\frac{3}{4}\sin 2\tau\sin^2 q - 6p^2\sin\tau\cos q\,.$$

$$K_1 = \frac{1}{2}p^2, \quad K_2 = 0, \quad K_3 = \frac{3}{2}\sin^2 q + 6\gamma\cos q + \frac{3(\alpha - \beta\cos q)^2}{\sin^2 q} \tag{1.11}$$

Simultaneously with the transformation of the Hamiltonian, we shall seek a corresponding canonical replacement of variables having the form

$$x = q + \varepsilon q^{(1)} + \frac{1}{2!}\varepsilon^2 q^{(2)} + \frac{1}{3!}\varepsilon^{(3)} q^{(3)} + O(\varepsilon^4)$$

$$X = p + \varepsilon p^{(1)} + \frac{1}{2!}\varepsilon^3 p^{(2)} + \frac{1}{3!}\varepsilon^2 p^{(3)} + O(\varepsilon^4)$$

The functions $q^{(i)}(q, p, \tau)$ and $p^{(i)}(q, p, \tau)$ ($i = 1, 2, 3$) are obtained using the formulae of the Depry–Hori method [10] (which are not derived here) using the expressions for W_i ($i = 1, 2, 3$) from (1.11). These functions have the form

$$q^{(1)} = 0, \quad q^{(2)} = 2\cos\tau\sin q, \quad q^{(3)} = -12p\sin\tau\cos q$$

$$p^{(1)} = 0, \quad p^{(2)} = -2p\cos\tau\cos q, \quad p^{(3)} = -0.75\sin 2\tau\sin 2q - 6p^2\sin\tau\sin q$$

After substituting the functions K_i from (1.11) into (1.10) we make one more canonical univalent replacement of variables $q, p \to u, \upsilon$, given by the formulae $u = \cos q$, $p = -\upsilon\sin q$, which reduce (1.1) to algebraic form. We have

$$K = \varepsilon\upsilon^2(1 - u^2)/2 + \varepsilon^3\Pi(u) + O(\varepsilon^4)$$

$$\Pi(u) = \frac{1}{4}(1 - u^2) + \gamma u + \frac{(\alpha - \beta u)^2}{2(1 - u^2)} \tag{1.12}$$

The equations of motion corresponding to (1.12) have the form

$$\frac{du}{d\tau}=\frac{\partial K}{\partial v},\quad \frac{dv}{d\tau}=-\frac{\partial K}{\partial u} \tag{1.13}$$

2. THE APPROXIMATE SYSTEM AND ITS EQUILIBRIUM POSITIONS

If we neglect terms $O(\varepsilon^4)$, the following autonomous system of differential equations will correspond to the truncated Hamiltonian obtained

$$\frac{du}{d\tau}=\varepsilon v(1-u^2),\quad \frac{dv}{d\tau}=\varepsilon u v^2-\varepsilon^3\frac{d\Pi}{du} \tag{2.1}$$

We will obtain the equilibrium positions of approximate system (2.1). Since $u \neq \pm 1$, in the equilibrium position $v = 0$, and the quantity u satisfies the relation $d\Pi/du = 0$, where $d\Pi/du = f(u) - u/2 + \gamma$ and $f(u) = (\alpha - \beta u)(\alpha u - \beta)(1 - u^2)^{-2}$.

The equation

$$f(u)=\frac{1}{2}u-\gamma \tag{2.2}$$

which the equilibrium values of u satisfy, will be investigated graphically in the interval $(-1, 1)$; its roots will be the abscissae of the points of intersection of the curve $y = f(u)$ and the straight line $y = u/2 - \gamma$.

As an analysis shows, the function $y = f(u)$ increases monotonically in the interval $(-1, 1)$ for any admissible values of α and β. Its derivative

$$f'(u)=\frac{(\alpha^2+\beta^2)(1+3u^2)-2\alpha\beta u(3+u^2)}{(1-u^2)^3} \tag{2.3}$$

has a minimum at $u = u_*$, which is the root of the equation $f'(u) = 0$, where

$$f''(u)=\frac{6[2(\alpha^2+\beta^2)u(1+u^2)-\alpha\beta(1+6u^2+u^4)]}{(1-u^2)^4}$$

where $u_* \geqslant 0$ when $\alpha\beta \geqslant 0$ and $u_* < 0$ when $\alpha\beta < 0$. When $-1 < u < u_*$ the function $y = f'(u)$ decreases monotonically, and when $u_* < u < 1$ it increases monotonically. Graphs of the functions $y = f(u)$ and $y = f'(u)$ are shown in Fig. 1 for the case when $\alpha\beta > 0$. The curve $y = f(u)$ intersects the ordinate axis at the point $(0, -\alpha\beta)$, while the curve $y = f'(u)$ intersects the ordinate axis at the point $(0, \alpha^2 + \beta^2)$.

The following cases of the intersection of the curve $y = f(u)$ and the straight line $y = u/2 - \gamma$ are possible.

Case 1. If $f'(u) > 1/2$ for all $u \in (-1, 1)$ (Fig. 1a), the curve $y = f(u)$ at each point will be "steeper" than the straight line $y = u/2 - \gamma$, which has a constant slope. For any value of the parameter $\gamma (\gamma > 0)$ the graphs of the functions $y = f(u)$ and $y = u/2 - \gamma$ intersect at a single point, the abscissa of which will henceforth be denoted by $\hat{u}$, and system (2.1) has a unique equilibrium position.

Case 2. If $f'(u^*) < 1/2$, the straight line $y = 1/2$ intersects the graph of the function $y = f'(u)$ at two points (Fig. 1b) and the equation $f'(u) = 1/2$ has two solutions, which we will denote by $u_{(1)}$ and $u_{(2)}$ $(u_{(1)} < u_{(2)})$. At points with abscissae $u = u_{(i)}$ $(i = 1, 2)$ the straight lines $y = u/2 - \gamma_{(i)}$, shown in Fig. 1(b) by the dash-dot lines, touch the curve $y = f(u)$; the quantities $\gamma_{(i)}$ $(i = 1, 2)$ are functions of the parameters α and β.

If $\gamma_{(1)} > 0$ and $\gamma_{(2)} > 0$, then for values of the parameter γ from the intervals $0 < \gamma < \gamma_{(1)}$ and $\gamma > \gamma_{(2)}$ the graphs of the functions $y = f(u)$ and $y = u/2 - \gamma$ intersect at a single point; we will denote its abscissa by $\hat{u}$. When $\gamma_{(1)} < \gamma < \gamma_{(2)}$ the graphs intersect at three points with abscissae $u = u_i$ $(i = 1, 2, 3)$, where $u_1 < u_{(1)} < u_2 < u_{(2)} < u_3$. In this case system (2.1) has one and three equilibrium positions respectively. If $\gamma = \gamma_{(1)}$ or $\gamma = \gamma_{(2)}$, the system has two equilibrium positions.

When $\gamma_{(1)} < 0$, $\gamma_{(2)} > 0$ we have three equilibrium positions if $0 < \gamma < \gamma_{(2)}$ and one equilibrium position if $\gamma > \gamma_{(2)}$; if $\gamma_{(1)} < 0$ and $\gamma_{(2)} < 0$, then, for all $\gamma > 0$ the system has one equilibrium position.